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Tight frame approximation for multi-frames and super-frames

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Abstract

We consider a generator $\Phi = (\phi_1, \dots, \phi_N)$ for either a multi-frame or a super-frame generated under the action of a projective unitary representation for a discrete countable group. Examples of such frames include Gabor multi-frames, Gabor super-frames and frames for shift-invariant subspaces. We show that there exists a unique normalized tight multi-frame (resp. super-frame) generator $\Psi = (\psi_1, \dots, \psi_N)$ such that $\sum_{j=1}^{N} ||\phi_j - \psi_j||^2 \leqslant \sum_{j=1}^{N} ||\phi_j - \psi_j||^2$ holds for all the normalized tight multi-frame (resp. super-frame) generators $\eta = (\eta_1, \dots, \eta_N)$. We also investigate the similar problems for dual frames and discuss a few applications to Gabor frames and some other frames.

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1. Introduction

For a given "basis" $\{x_n\}$ in a Hilbert space, it has been an interesting question how to get a "nice" basis $\{y_n\}$ which is close to the given $\{x_n\}$ and generates the

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same subspace. In the case that $\{x_n\}$ is linearly independent, a well-known approach is the Gram–Schmidt orthonormalization process. This approach is inherently order-dependent in that a reordering of $\{x_n\}$ will generally result in an entirely new orthonormal set. This order-dependent character may not be desirable in some applications (cf. [FPT]). Moreover, the Gram–Schmidt process also fails when the given "basis" has redundancy property (such a "basis" is called a frame). All these considerations lead us to seek a different approach which should be order-independent and also valid for redundancy bases. One such approach is the so-called *symmetric approximation* by normalized tight frames recently introduced by Frank et al. [FPT] for redundancy bases. When $\{x_n\}$ is a linearly independent set, this symmetric approximation is also called *Löwdin orthogonalization* (cf. [FPT,AEG1,AEG2,GL,Lo]).

In applications we are more interested in those frames with special structures (e.g. wavelet frames, Gabor frames, frames for shift invariant spaces). So when we consider tight frame approximation, it is natural to require the tight frame to be of the same kind. Note that Gabor frames, wavelet frames and many other interesting frames are generated by a collection of unitary transformations and some (single or multi) window functions. In all these situations, the symmetric approximation fails to work when the underlying Hilbert space is infinite dimensional (see [Han,JS]). Instead of using the symmetric approximations, we approximate the frame generator by normalized tight frame generators when the underlying frame is generated by a collection of unitary operators. This leads to the natural question: When do we have a best normalized tight frame approximation for such frames? The existence and uniqueness result for such a best approximation was proved in [Han] for frames which are generated by a single element generator and by a projective unitary representation of a countable group. This class of frames includes Gabor frames (for arbitrary lattices and any dimensions) and any frames induced by a group action such as frames for shift invariant subspaces. Independently, Janssen and Strohmer [JS] established the same result for Gabor frames in one-dimensional case. However, the main technique used in [Han] fails to work for multi-frames (See Example 1.1). The purpose of the present paper is to use a different (more direct) approach to investigate the tight frame approximation for frames with multi generators. We will also investigate the tight frame approximations for super frames introduced by Balan, Han and Larson ([Ba,HL]). To state the problems and the results, we need to recall some notations and definitions.

A frame for a separable Hilbert space $\mathcal{H}$ is a sequence $\{x_n\}$ in $\mathcal{H}$ such that there exist $A, B > 0$ with the property that

$$A||x||^2 \leqslant \sum_n |\langle x, x_n \rangle|^2 \leqslant B||x||^2 \quad (1)$$

holds for all $x \in \mathcal{H}$. The optimal constants (maximal for A and minimal for B) are called frame bounds. When $A = B = 1$, $\{x_n\}$ is called a *normalized tight frame* (*or Parseval frame*). A sequence $\{x_n\}$ is called *Bessel* if we only require the right side inequality of (1) to hold. In order to introduce the concept of super-frames, we also need the notion of *strong disjointness of frames* which was formally introduced in

[HL]: Two Bessel sequences $\{x_n\}$ and $\{y_n\}$ are called *strongly disjoint* if

$$\sum_n \langle x, x_n \rangle y_n = 0$$

holds for all $x \in \mathcal{H}$.

For each frame $\{x_n\}$ there exists a *standard dual frame* $\{S^{-1}x_n\}$, which together with the frame $\{x_n\}$ provides a "reconstruction" formula for elements in $\mathcal{H}$:

$$x = \sum_n \langle x, S^{-1}x_n \rangle x_n, \quad x \in \mathcal{H}. \tag{2}$$

where S is the positive invertible linear operator on $\mathcal{H}$ defined by

$$Sx = \sum_n \langle x, x_n \rangle x_n, \quad x \in \mathcal{H}.$$

This operator S is called the *frame operator* for $\{x_n\}$. From the definition of S, it follows immediately that $\{S^{-1/2}x_n\}$ is a normalized tight frame for $\mathcal{H}$. A frame $\{y_n\}$ is called a *dual* for $\{x_n\}$ if (1) holds when $S^{-1}x_n$ is replaced by y_n. We remark that if a frame is not a Riesz basis, then it has infinitely many duals.

The symmetric approximation investigated by Frank et al. [FPT] can be phrased as the following: Let $\{x_n\}$ be a frame for $\mathcal{H}$. A normalized tight frame $\{y_n\}$ for H is said to be a *symmetric approximation* of $\{x_n\}$ if the inequality

$$\sum_n ||z_n - x_n||^2 \geqslant \sum_n ||y_n - x_n||^2 \tag{3}$$

is valid for all normalized tight frames $\{z_n\}$ of H.

Many interesting frames are generated by some (usually finite number of) "window" functions under the action of a collection of unitary operators. For example, Gabor frames and wavelet frames are of this kind. For convenience, we call a countable collection $\mathcal{U}$ of unitary operators a *unitary system* if it contains the identity operator. For $\Phi = (\phi_1, \dots, \phi_N)$ with $\phi_j \in \mathcal{H}$, if $\{U\phi_j : U \in \mathcal{U}, 1 \leqslant j \leqslant N\}$ is a frame (resp. normalized tight frame) for $\mathcal{H}$, then we call Φ a *multi-frame generator* (resp. *normalized tight multi-frame generator*) of length N for $\mathcal{U}$. Similarly, Φ is called a Bessel sequence generator if $\mathcal{U}\Phi$ is a Bessel sequence.

In the normalized tight frame approximation, if we restrict ourselves to the frames induced by a unitary system, then the symmetric approximation is not a good choice since the summation in (3) is always infinite if the given frame is not normalized tight. In this case we use the natural metric: Let $\Phi = (\phi_1, \dots, \phi_N)$ be a multi-frame generator for a unitary system $\mathcal{U}$. Then a normalized tight multi-frame $\Psi = (\psi_1, \dots, \psi_N)$ for $\mathcal{U}$ is called a *best normalized tight multi-frame approximation* for Φ if the inequality

$$\sum_{k=1}^{N} ||\phi_k - \psi_k||^2 \leqslant \sum_{k=1}^{N} ||\phi_k - \xi_k||^2 \tag{4}$$

is valid for all the normalized tight multi-frame generator $\xi = (\xi_1, \dots, \xi_N)$ for $\mathcal{U}$. We remark that it is not hard to check that if Ψ is a best normalized tight multi-frame approximation for Φ, then

$$\sum_{k=1}^{N} ||\phi_k - \psi_k||^2 = \min\left\{\sum_{k=1}^{N} ||\phi_k - \xi_{\sigma(k)}||^2 : \xi, \sigma\right\},$$

where the minimum is taken over all the normalized tight multi-frame generators $\xi = (\xi_1, \dots, \xi_N)$ for $\mathcal{U}$ and all the permutation σ of $\{1, 2, \dots, N\}$.

For a general unitary system $\mathcal{U}$, the best normalized tight multi-frame generators may not even exist (see [Han]). In this paper we continue to focus our investigation on a nice class of unitary systems: *group-like unitary systems* [GH1]. This class contains many interesting examples including unitary group systems and Gabor systems for arbitrary lattices (see Section 4 for definitions).

Group-like unitary systems are simply the images of projective unitary representations for countable discrete groups. Recall that a *projective unitary representation* π for a countable discrete (not necessarily abelian) group $\mathcal{G}$ is a mapping $g \to U_g$ from $\mathcal{G}$ into the set of unitary operators on a Hilbert space $\mathcal{H}$ such that $U_g U_h = \mu(g, h) U_{gh}$ for all $g, h \in \mathcal{G}$, where $\mu(g, h)$ belongs to the circle group $\mathbb{T}$ (cf. [Va]). In general for a countable set of unitary operators $\mathcal{U}$ acting on a separable Hilbert space H which contains the identity operator, we will call $\mathcal{U}$ *group-like* if

$$group(\mathcal{U}) \subset \mathbb{T}\mathcal{U} := \{tU : \ t \in \mathbb{T}, \ U \in \mathcal{U}\}$$

and if different U and V in $\mathcal{U}$ are always linearly independent, where $group(\mathcal{U})$ denotes the group generated by $\mathcal{U}$ with respect to multiplication. A group-like unitary system $\mathcal{U}$ is always an image of a projective unitary representation π for the group $\mathcal{G} := group(\mathcal{U})$ (see [Han]). For singly-generated frame $\{U\phi : U \in \mathcal{U}\}$, we have the following:

Theorem 1.1 ([Han]). *Let $\mathcal{U}$ be a group-like unitary system acting on a Hilbert space $\mathcal{H}$ and let ϕ be a frame generator for $\mathcal{U}$. Then $S^{-1/2}\phi$ is the unique best normalized tight frame approximation for ϕ, where S is the frame operator for the frame $\{U\phi : U \in \mathcal{U}\}$.*

A crucial ingredient in the proof of the above theorem is the following parametrization result for all the normalized tight frame generators in terms of the unitary operators in the von Neumann algebra generated by the system $\mathcal{U}$:

Theorem 1.2. *Let $\mathcal{U}$ be a group-like unitary system acting on a Hilbert space $\mathcal{H}$ and ϕ be a normalized tight frame generator for $\mathcal{U}$. Then $\eta \in \mathcal{H}$ is a normalized tight frame generator for $\mathcal{U}$ if and only if there exits a unitary operator $A \in w^*(\mathcal{U})$ such that $\eta = A\phi$, where $w^*(\mathcal{U})$ is the von Neumann algebra generated by $\mathcal{U}$.*

However, such a parametrization result is no longer valid for multi-frames (see Example 1.1 below). Therefore the approach in [Han] cannot be applied to the multi-frame case. In Section 2 we will generalize Theorem 1.1 to multi-frame generators and provide a different approach to this generalization. This new proof is much more elementary and transparent. Following the same line we will examine the distance between a frame and its duals, and we will also give a best normalized tight frame approximation result for super-frames. In Section 3 we discuss a few applications of our results to Gabor frames and frames for shift invariant subspaces.

Example 1.1. Let $H = L^2[0,1]$ and $\mathcal{U} = \{M_{e^{2\pi int}} : n \in \mathbb{Z}\}$, where M_h denotes the multiplication operator by symbol h. Let $\Phi = (\chi_{[0,1/2)}, \chi_{[1/2,1]})$ and $\Psi = (\chi_{[0,1/4)}, \chi_{[1/4,1]})$. Then both Ψ and Φ are normalized tight multi-frame generators (of length 2). However there is NO unitary operator U on H which maps $\chi_{[0,1/2)}$ to either $\chi_{[0,1/4)}$ or $\chi_{[1/4,1]}$ since unitary operators preserve vector norm.

2. Approximation for multi-frames and super-frames

We first generalize Theorem 1.1 to the multi-frame case. Let Φ be a Bessel sequence generator for a unitary system $\mathcal{U}$. We use T_Φ to denote the *analysis operator* from $\mathcal{H}$ to $\mathcal{L}^2(\mathcal{U} \times \{1, \dots, N\})$ defined by:

$$T_\Phi x = \sum_{j=1}^{N} \sum_{U \in \mathcal{U}} \langle x, U\phi_j \rangle e(U,j), \quad x \in \mathcal{H},$$

where $\{e(U,j) : U \in \mathcal{U}, 1 \leqslant j \leqslant N\}$ is the standard orthonormal basis for $\mathcal{L}^2(\mathcal{U} \times \{1, \dots, N\})$. Then the adjoint operator of T_Φ is the *synthesis operator* satisfying:

$$T_\Phi^* e(U,j) = U\phi_j, \quad U \in \mathcal{U}, j \in \{1, \dots, N\}.$$

Lemma 2.1. *Let $\mathcal{U}$ be a group-like unitary system on $\mathcal{H}$.*

(i) *If $\Phi = (\phi_1, \dots, \phi_N)$ is a normalized tight multi-frame generator for $\mathcal{U}$, then it is also a normalized tight multi-frame generator for the group-like unitary system $\mathcal{U}^*$, where $\mathcal{U}^* = \{U^* : U \in \mathcal{U}\}$.*
(ii) *Suppose that $\xi = (\xi_1, \dots, \xi_N)$ and $\eta = (\eta_1, \dots, \eta_N)$ are two Bessel sequence generators for $\mathcal{U}$. Then*

$$\sum_{j=1}^{N} \sum_{U \in \mathcal{U}} \langle \phi_k, U^*\xi_j \rangle \langle U^*\eta_j, \phi_k \rangle = \sum_{j=1}^{N} \sum_{U \in \mathcal{U}} \langle \phi_k, U\xi_j \rangle \langle U\eta_j, \phi_k \rangle.$$

Proof. This follows immediately from the definition of group-like unitary systems. □

Lemma 2.2. *Let $\mathcal{U}$ be a group-like unitary system on $\mathcal{H}$.*

(i) *Suppose that $\xi = (\xi_1, \dots, \xi_N)$ and $\eta = (\eta_1, \dots, \eta_N)$ are two Bessel sequence generators such that $\{U\xi_j : U \in \mathcal{U}, 1 \leqslant j \leqslant N\}$ and $\{U\eta_j : U \in \mathcal{U}, 1 \leqslant j \leqslant N\}$ are strongly disjoint. Then $\sum_{j=1}^{N} \langle \eta_j, \xi_j \rangle = 0$.*

(ii) *Suppose that* $\{U\phi_j : U\in\mathcal{U}, 1\leqslant j\leqslant N\}$ *is a dual of* $\{U\psi_j : U\in\mathcal{U}, 1\leqslant j\leqslant N\}$, *and* $\{U\xi_j : U\in\mathcal{U}, 1\leqslant j\leqslant N\}$ *is a dual of* $\{U\eta_j : U\in\mathcal{U}, 1\leqslant j\leqslant N\}$. *Then*

$$\sum_{j=1}^{N} \langle \phi_j, \psi_j \rangle = \sum_{k=1}^{L} \langle \xi_k, \eta_k \rangle.$$

In particular if $\Phi = (\phi_1, \dots, \phi_N)$ *and* $\eta = (\eta_1, \dots, \eta_L)$ *are two normalized tight multi-frame generators for a group-like unitary system* $\mathcal{U}$. *Then* $\sum_{j=1}^{N}||\phi_j||^2 = \sum_{k=1}^{L}||\eta_k||^2$.

Proof.

(i) By Theorem 2 in [GH2], there exists $\Phi = (\phi_1, \dots, \phi_K)$ such that $\{U\phi_j : U\in\mathcal{U},\ j = 1, \dots, K\}$ is a normalized tight frame generator of $\mathcal{H}$. Thus

$$\begin{aligned}
\sum_{j=1}^{N} \langle \eta_j, \xi_j \rangle &= \sum_{j=1}^{N} \sum_{k=1}^{K} \sum_{U\in\mathcal{U}} \langle \eta_j, U\phi_k \rangle \langle U\phi_k, \xi_j \rangle \\
&= \sum_{k=1}^{K} \sum_{j=1}^{N} \sum_{U\in\mathcal{U}} \langle U^*\eta_j, \phi_k \rangle \langle \phi_k, U^*\xi_j \rangle \\
&= \sum_{k=1}^{K} \left(\sum_{j=1}^{N} \sum_{U\in\mathcal{U}} \langle \phi_k, U^*\xi_j \rangle \langle U^*\eta_j, \phi_k \rangle \right) \\
&= \sum_{k=1}^{K} \left(\sum_{j=1}^{N} \sum_{U\in\mathcal{U}} \langle \phi_k, U\xi_j \rangle \langle U\eta_j, \phi_k \rangle \right) \\
&= 0,
\end{aligned}$$

where we use Lemma 2.1(ii) in the fourth equality and the strong disjointness in the last equality.

(ii) can be checked in a similar way. □

Theorem 2.3. *Let* $\mathcal{U}$ *be a group-like unitary system acting on a Hilbert space* $\mathcal{H}$ *and let* $\Phi = (\phi_1, \dots, \phi_N)$ *be a multi-frame generator for* $\mathcal{U}$. *Then* $S^{-1/2}\Phi$ *is the unique best normalized tight multi-frame approximation for* Φ, *where* S *is the frame operator for the multi-frame* $\{U\phi_j : U\in\mathcal{U},\ j = 1, \dots, N\}$.

Proof. It is a routine exercise to check that $SU = US$ for all $U\in\mathcal{U}$ (cf. the proof of Theorem 1.2 in [Han] for the one generator case). Thus implies that both $S^{-1/2}, S^{-1/4}$ also commute with every element in $\mathcal{U}$.

Now let $\Psi = \{\psi_1, \dots, \psi_N\}$ be any normalized tight multi-frame generator for $\mathcal{U}$. We first prove that

$$\sum_{k=1}^{N} \langle T_{\Psi}^{*} T_{S^{-1/2}\Phi} S^{-1/4}\phi_k, S^{-1/4}\phi_k \rangle = \sum_{k=1}^{N} \langle \psi_k, \phi_k \rangle. \tag{5}$$

In fact, by the definition of analysis operator we have that the left side of (5) is equal to:

$$\begin{aligned}
&\sum_{k=1}^{N} \left\langle \sum_{j=1}^{N} \sum_{U\in\mathcal{U}} \langle S^{-1/4}\phi_k, US^{-1/2}\phi_j \rangle U\psi_j, S^{-1/4}\phi_k \right\rangle \\
&= \sum_{k=1}^{N} \sum_{j=1}^{N} \sum_{U\in\mathcal{U}} \langle S^{-1/4}\phi_k, US^{-1/2}\phi_j \rangle \langle U\psi_j, S^{-1/4}\phi_k \rangle \\
&= \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{U\in\mathcal{U}} \langle S^{-1/4}U^*\phi_k, S^{-1/2}\phi_j \rangle \langle \psi_j, S^{-1/4}U^*\phi_k \rangle \\
&= \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{U\in\mathcal{U}} \langle S^{-1/4}\psi_j, U^*\phi_k \rangle \langle U^*\phi_k, S^{-3/4}\phi_j \rangle \\
&= \sum_{j=1}^{N} \langle SS^{-1/4}\psi_j, S^{-3/4}\phi_j \rangle \\
&= \sum_{j=1}^{N} \langle \psi_j, \phi_j \rangle .
\end{aligned}$$

Since $\{U\psi_j : U\in\mathcal{U},\ j=1,\dots,N\}$ and $\{US^{-1/2}\phi_j : U\in\mathcal{U},\ j=1,\dots,N\}$ are normalized tight frames, we have that $||T_{\Psi}^*|| = ||T_{S^{-1/2}\Phi}|| = 1$. Therefore, from (5), we have

$$\begin{aligned}
\left| \sum_{k=1}^{N} \langle \psi_k, \phi_k \rangle \right| &\leqslant \sum_{k=1}^{N} |\langle T_{\Psi}^* T_{S^{-1/2}\Phi} S^{-1/4}\phi_k, S^{-1/4}\phi_k \rangle| \\
&\leqslant \sum_{k=1}^{N} |||T_{\Psi}^* T_{S^{-1/2}\Phi} S^{-1/4}\phi_k||\ ||S^{-1/4}\phi_k|| \\
&\leqslant \sum_{k=1}^{N} ||S^{-1/4}\phi_k||^2 \\
&= \sum_{k=1}^{N} \langle \phi_k, S^{-1/2}\phi_k \rangle .
\end{aligned}$$

Hence from Lemma 2.2(ii) and the above inequality we have

$$\begin{aligned}
\sum_{k=1}^{N} ||\phi_k - \psi_k||^2 &= \sum_{k=1}^{N} ||\phi_k||^2 + \sum_{k=1}^{N} ||\psi_k||^2 - 2Re\langle \phi_k, \psi_k \rangle \\
&= \sum_{k=1}^{N} ||\phi_k||^2 + \sum_{k=1}^{N} ||S^{-1/2}\phi_k||^2 - 2\sum_{k=1}^{N} Re\langle \phi_k, \psi_k \rangle
\end{aligned}$$

$$\geqslant \sum_{k=1}^{N} ||\phi_k||^2 + \sum_{k=1}^{N} ||S^{-1/2}\phi_k||^2 - 2\sum_{k=1}^{N} \langle \phi_k, S^{-1/2}\phi_k \rangle$$
$$= \sum_{k=1}^{N} ||\phi_k - S^{-1/2}\phi_k||^2.$$

This implies that $S^{-1/2}\Phi$ is the best normalized tight multi-frame approximation for Φ.

Now assume that $\xi = (\xi_1, \dots, \xi_N)$ is another best normalized tight multi-frame approximation for Φ. Then, we have

$$\sum_{k=1}^{N} ||\xi_k - \phi_k||^2 = \sum_{k=1}^{N} ||S^{-1/2}\phi_k - \phi_k||^2$$

which implies that $Re\sum_{k=1}^{N} \langle \xi_k, \phi_k \rangle = \sum_{k=1}^{N} ||S^{-1/4}\phi_k||^2$ by Lemma 2.2(ii).

Write $S^{-1/4}\Phi = (S^{-1/4}\phi_1, \dots, S^{-1/4}\phi_N)$ and consider ξ, Φ and $S^{-1/4}\Phi$ as vectors in the direct sum Hilbert space $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$. Then we have

$$Re\langle \xi, \Phi \rangle = Re\langle S^{1/4}\xi, S^{-1/4}\Phi \rangle = ||S^{-1/4}\Phi||^2. \tag{6}$$

However,

$$\begin{aligned}
||S^{1/4}\xi||^2 &= \sum_{k=1}^{N} ||S^{1/4}\xi_k||^2 \\
&= \sum_{k=1}^{N} \sum_{j=1}^{N} \sum_{U \in \mathcal{U}} |\langle S^{1/4}\xi_k, US^{-1/2}\phi_j \rangle|^2 \\
&= \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{U \in \mathcal{U}} |\langle U^*\xi_k, S^{-1/4}\phi_j \rangle|^2 \\
&= \sum_{j=1}^{N} ||S^{-1/4}\phi_j||^2 = ||S^{-1/4}\Phi||^2.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
|\langle S^{1/4}\xi, S^{-1/4}\Phi \rangle| &= |\langle \xi, \Phi \rangle| \geqslant Re\langle \xi, \Phi \rangle = Re\langle S^{1/4}\xi, S^{-1/4}\Phi \rangle \\
&= ||S^{-1/4}\Phi||^2 = ||S^{1/4}\xi|| \; ||S^{-1/4}\Phi||,
\end{aligned}$$

this implies by the Cauchy–Schwarz inequality that

$$|\langle S^{1/4}\xi, S^{-1/4}\Phi \rangle| = ||S^{1/4}\xi|| \; ||S^{-1/4}\Phi||.$$

Thus there is $\lambda \in \mathbb{C}$ which implies that $|\lambda| = 1$ and $S^{1/4}\xi = \lambda S^{-1/4}\Phi$. Therefore $\xi = \lambda S^{-1/2}\Phi$. From $\langle S^{1/4}\xi, S^{-1/4}\Phi \rangle = |\langle S^{1/4}\xi, S^{-1/4}\Phi \rangle|$, it follows that $\lambda = 1$. Hence $\xi = S^{-1/2}\Phi$, as expected. $\square$

Remark. We note that $\mathcal{U}S^{-1/2}\Phi$ is actually also the best normalized tight multi-frame approximation simultaneously for all the frames $\mathcal{U}\Phi_\alpha$ $(\alpha\in\mathbb{R})$, where $\Phi_\alpha = (S^\alpha\phi_1, \dots, S^\alpha\phi_N)$. Indeed, this follows from the fact that $S^{2\alpha+1}$ is the frame operator for frame $\mathcal{U}\Phi_\alpha$ and $(S^{2\alpha+1})^{-1/2}\Phi_\alpha = S^{-1/2}\Phi$.

We can also use Lemma 2.2 to examine the minimization problem between a frame and all of its duals. Given a multi-frame generator $\Phi = (\phi_1, \dots, \phi_N)$ for a group-like unitary system $\mathcal{U}$, then $S^{-1}\Phi := (S^{-1}\phi_1, \dots, S^{-1}\phi_N)$ generates the standard dual of the frame $\mathcal{U}\Phi$. Standard duals have several nice features over the alternate duals. For example, it is well-known (cf. [DLL]) that a dual $\mathcal{U}\eta$ with $\Psi = (\eta_1, \dots, \eta_N)$ is the standard dual if and only if $\sum_{j=1}^N \sum_{U\in\mathcal{U}} |\langle x, U\eta_j\rangle|^2 \leqslant \sum_{j=1}^N \sum_{U\in\mathcal{U}} |\langle x, U\xi_j\rangle|^2$ holds for all $x\in\mathcal{H}$ and for any dual $\mathcal{U}\xi$ with $\xi = (\xi_1, \dots, \xi_N)$. In the next result we prove that the standard dual also minimizes its "distance" to the frame over all the other duals. A normalized version for single generator Gabor frames is well known (cf. [Ja]): Let g be a Gabor frame generator in $L^2(\mathbb{R})$. Then the canonical dual $S^{-1}g$ minimizes

$$\left|\left|\frac{g}{||g||} - \frac{\gamma}{||\gamma||}\right|\right|$$

over all dual windows γ. The following theorem tells us that this is also true for the non-normalized case, for multi-windows and for arbitrary group-like unitary systems.

Theorem 2.4. *Let $\Phi = (\phi_1, \dots, \phi_N)$ be a multi-frame generator for a group-like unitary system $\mathcal{U}$, and $\eta = (\eta_1, \dots, \eta_N)$ be a dual frame generator for $\mathcal{U}\Phi$. Then the following are equivalent*:

(i) *$\mathcal{U}\eta$ is the standard dual of $\mathcal{U}\Phi$, i.e. $\eta_j = S^{-1}\phi_j (j = 1, \dots, N)$, where S is the frame operator for the frame $\{U\phi_j : U\in\mathcal{U},\ j = 1, \dots, N\}$.*
(ii) $\sum_{j=1}^N ||\eta_j||^2 = \sum_{j=1}^N ||S^{-1}\phi_j||^2$
(iii) $\sum_{j=1}^N ||\eta_j - \phi_j||^2 \leqslant \sum_{j=1}^N ||\xi_j - \phi_j||^2$ *holds for any dual frame generator* $\xi = (\xi_1, \xi_2, \dots, \xi_N)$.

Proof. We first prove the equivalence between (i) and (ii). Clearly, (i) $\Rightarrow$ (ii). Now assume (ii) holds. Since both η and $S^{-1}\Phi$ are dual frame generators, it follows from the definition of duals that $\mathcal{U}(\eta - S^{-1}\Phi)$ and $\mathcal{U}S^{-1}\Phi$ are strongly disjoint Bessel sequences. Thus, by Lemma 2.2(i), we have that

$$\sum_{j=1}^N ||\eta_j||^2 = \sum_{j=1}^N ||S^{-1}\phi_j + (\eta_j - S^{-1}\phi_j)||^2$$

$$
\begin{aligned}
&= \sum_{j=1}^{N} ||S^{-1}\phi_j||^2 + \sum_{j=1}^{N} ||(\eta_j - S^{-1}\phi_j)||^2 \\
&\quad + 2Re\sum_{j=1}^{N} \langle S^{-1}\phi_j, \eta_j - S^{-1}\phi_j \rangle \\
&= \sum_{j=1}^{N} ||S^{-1}\phi_j||^2 + \sum_{j=1}^{N} ||(\eta_j - S^{-1}\phi_j)||^2.
\end{aligned}
$$

Therefore condition (ii) implies that $\eta_j = S^{-1}\phi_j$.

Suppose that (ii) holds. We check for (iii). Since (ii) implies (i), we have that $\eta = S^{-1}\Phi$. Let ξ be any dual frame generator for $\mathcal{U}\Phi$. Then from the above argument we have that

$$
\sum_{j=1}^{N} ||S^{-1}\phi_j||^2 \leqslant \sum_{j=1}^{N} ||\xi_j||^2.
$$

Thus

$$
\begin{aligned}
\sum_{j=1}^{N} ||\eta_j - \phi_j||^2 &= \sum_{j=1}^{N} ||S^{-1}\phi_j - \phi_j||^2 \\
&= \sum_{j=1}^{N} ||S^{-1}\phi_j||^2 + \sum_{j=1}^{N} ||\phi_j||^2 - 2Re\sum_{j=1}^{N} \langle S^{-1}\phi_j, \phi_j \rangle \\
&\leqslant \sum_{j=1}^{N} ||\xi_j||^2 + \sum_{j=1}^{N} ||\phi_j||^2 - 2Re\sum_{j=1}^{N} \langle S^{-1}\phi_j, \phi_j \rangle.
\end{aligned}
$$

From Lemma 2.2(ii), we have $\sum_{j=1}^{N} \langle \xi_j, \phi_j \rangle = \sum_{j=1}^{N} \langle S^{-1}\phi_j, \phi_j \rangle$. Therefore

$$
\begin{aligned}
\sum_{j=1}^{N} ||\eta_j - \phi_j||^2 &\leqslant \sum_{j=1}^{N} ||\xi_j||^2 + \sum_{j=1}^{N} ||\phi_j||^2 - 2Re\sum_{j=1}^{N} \langle \xi_j, \phi_j \rangle \\
&= \sum_{j=1}^{N} ||\xi_j - \phi_j||^2.
\end{aligned}
$$

Finally we assume that (iii) holds. Then from the above argument we have

$$
\sum_{j=1}^{N} ||\eta_j - \phi_j||^2 = \sum_{j=1}^{N} ||S^{-1}\phi_j - \phi_j||^2.
$$

Thus applying Lemma 2.2(ii) again, we obtain $\sum_{j=1}^{N} ||\eta_j||^2 = \sum_{j=1}^{N} ||S^{-1}\phi_j||^2$. □

At the end of this section we examine the normalized tight frame approximation for super-frames. Super-frames (or *disjoint frames*) were formally introduced by Balan [Ba], Han and Larson [HL] and were extensively studied in those two papers. Although the definition of super-frames is for general frames, here we restrict ourselves to the unitary system generated frames. Let $\phi_1, \ldots, \phi_N \in \mathcal{H}$. If $\{U\phi_1 \oplus \cdots \oplus U\phi_N : U \in \mathcal{U}\}$ is a frame for the orthogonal direct sum space (super-space) $\mathcal{H}^{(N)} := \mathcal{H} \oplus \cdots \oplus \mathcal{H}$, then we say that $\Phi = (\phi_1, \ldots, \phi_N)$ is a *super-frame generator*. It is a trivial fact that if Φ is a super-frame generator, then for each $j, \mathcal{U}\phi_j$ is a frame for $\mathcal{H}$. Clearly, the converse is not true.

An interesting special case is when the super-frame is composed of *strongly disjoint frames* $\mathcal{U}\phi_1, \ldots, \mathcal{U}\phi_N$. In this case we have

$$\sum_{U \in \mathcal{U}} \langle x, U\phi_j \rangle U\phi_k = 0, \quad x \in \mathcal{H},$$

holds when $j \neq k$. We remark that not every super-frame $(\phi_1, \ldots, \phi_N)$ is composed of strongly disjoint frames (see [HL]). The following is immediate from Theorem 1.1 (or Theorem 2.3):

Theorem 2.5. *Let $\Phi = (\phi_1, \ldots, \phi_N)$ be a super-frame generator for $\mathcal{U}$ and S be its frame operator (acting on the direct sum Hilbert space $\mathcal{H}^{(N)}$). Let $\eta := (\eta_1, \ldots, \eta_N) = S^{-1/2}\Phi \in \mathcal{H}^N$. Then η is the unique best normalized tight super-frame approximation for Φ.*

For a super-frame $(\phi_1, \ldots, \phi_N)$ we would also expect that $(S_1^{-1/2}\phi_1, \ldots, S_N^{-1/2}\phi_N)$ is a best normalized tight super-frame generator approximation for $(\phi_1, \ldots, \phi_N)$, where S_j is the frame operator for frame $\mathcal{U}\phi_j$. However this is not true in general since $(S_1^{-1/2}\phi_1, \ldots, S_N^{-1/2}\phi_N)$ is not necessarily a normalized tight super-frame generator. Indeed we have the following:

Theorem 2.6. *Let $(\phi_1, \ldots, \phi_N)$ be a super-frame generator for $\mathcal{U}$. Then the following are equivalent*

(i) $(S_1^{-1/2}\phi_1, \ldots, S_N^{-1/2}\phi_N)$ *is a best normalized tight super-frame generator approximation for* $(\phi_1, \ldots, \phi_N)$.
(ii) $(S_1^{-1/2}\phi_1, \ldots, S_N^{-1/2}\phi_N)$ *is a normalized tight super-frame generator.*
(iii) $\{\mathcal{U}\phi_1, \ldots, \mathcal{U}\phi_N\}$ *is a strongly disjoint* N-tuple.

Proof. (i) $\Rightarrow$ (ii) is obvious. For (ii) $\Rightarrow$ (iii) we refer to Theorem 2.9 in [HL]. Now we check (iii) $\Rightarrow$ (i). Let $\mathcal{U}^N = \{U^{(N)} = U \oplus \cdots \oplus U : U \in \mathcal{U}\}$ be the group-like unitary system on the direct sum Hilbert space $\mathcal{H}^N := \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ and $\Phi = \phi_1 \oplus \cdots \oplus \phi_N \in \mathcal{H}^N$. Then Φ is a frame generator for $\mathcal{U}^N$. From Theorem 1.1 we have that $S^{-1/2}\Phi$ is a best normalized tight frame generator approximation for Φ, where S is the frame operator of $\mathcal{U}^N\Phi$. Write $S = (S_{ij})_{N\times N}$ with S_{ij} being bounded linear operator on $\mathcal{H}$. Then the strong disjointness of $\{\mathcal{U}\phi_1, \ldots, \mathcal{U}\phi_N\}$ implies that

$S_{ij} = 0$ when $i \neq j$ and $S_{ii} = S_i$. Thus $S^{-1/2} = S_1^{-1/2} \oplus \ldots \oplus S_N^{-1/2}$ and so $S^{-1/2}\Phi = (S_1^{-1/2}\phi_1, \ldots, S_N^{-1/2}\phi_N)$. $\square$

3. Some applications

3.1. Gabor multi-frames

Let Λ be a full-rank lattice in $\mathbb{R}^d \times \mathbb{R}^d$, and let $g(x) \in L^2(\mathbb{R}^d)$. The *Gabor family* associated with Λ and g is the collection:

$$\mathcal{G}(\Lambda, g) = \{e^{2\pi i\langle m,x\rangle} g(x-n), \quad (m,n) \in \Lambda\}.$$

Such a family was first introduced by Gabor [Ga] in 1946 for the purpose of signal processing. When $\mathcal{G}(\Lambda, g)$ is a frame for $L^2(\mathbb{R}^d)$, we call g a *Gabor frame generator*.

We define, for any $(s,t) \in \mathbb{R}^{d\times d}$, the translation and modulation unitary operators are defined by:

$$T_t f(x) = f(x-t)$$

and

$$E_s f(x) = e^{2\pi i\langle s,x\rangle} f(x)$$

for all $f \in L^2(\mathbb{R}^d)$. Then E_s and T_t are unitary operators on $L^2(\mathbb{R}^d)$. Write $\mathcal{U}_\Lambda = \{E_m T_n : (m,n) \in \Lambda\}$. We will call $\mathcal{U}_\Lambda$ *a Gabor unitary system*. It is a trivial exercise that $\mathcal{U}_\Lambda$ is a group-like unitary system.

In general, a single function Gabor frame generator does not exist. In fact, a necessary condition for the existence of a single function Gabor frame generator is that $|detA| \leqslant 1$, where A is a $2d \times 2d$ non-singular real matrix with $\Lambda = A\mathbb{Z}^{2d}$ (cf. [CDH,DLL,HW1,Rie,RS1,RSt] etc.). Although it is known that this condition is also sufficient for "most" of the lattices, it remains an open problem whether this is true in general (cf. [HW1,HW2]). However, for each lattice Λ we can consider multi-window generators for Gabor unitary systems: Let $g_j \in L^2(\mathbb{R}^d)$ $(j = 1, \ldots, N)$. If $\mathcal{G}(\Lambda, g_1) \bigcup \mathcal{G}(\Lambda, g_2) \bigcup \cdots \bigcup \mathcal{G}(\Lambda, g_N)$ is a frame for $L^2(\mathbb{R}^d)$, then $(g_1, \ldots, g_N)$ is called a Gabor multi-frame generator. Applying Theorem 2.3 to Gabor multi-frames we obtain

Corollary 3.1. *Let $\mathcal{G}(\Lambda, g_1) \bigcup \mathcal{G}(\Lambda, g_2) \bigcup \cdots \bigcup \mathcal{G}(\Lambda, g_N)$ be a Gabor multi-frame generator and S be the associated frame operator. Then $(S^{-1/2}g_1, \ldots, S^{-1/2}g_N)$ is the unique best normalized tight Gabor multi-frame generator for $(g_1, \ldots, g_N)$.*

For the single window $(N = 1)$ case, Theorem 3.1 was proved by Janssen and Strohmer in [JS] when $d = 1$ and $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$, and independently, it was proved in [Han] for arbitrary lattices and arbitrary d. Janssen and Strohmer's proof uses different representations of the Gabor frame operator S which is only available for special cases.

As a special case of Theorem 2.4, we also have

Corollary 3.2. *Let* $\mathcal{G}(\Lambda, g_1) \bigcup \mathcal{G}(\Lambda, g_2) \bigcup \cdots \bigcup \mathcal{G}(\Lambda, g_N)$ *be a Gabor multi-frame generator and S be the associated frame operator. Then*

$$\sum_{j=1}^{N} ||g_j - S^{-1/2} g_j||^2 \leqslant \sum_{j=1}^{n} ||g_j - h_j||^2$$

holds for all $(h_1, \ldots, h_N)$ *such that*

$$f = \sum_{j=1}^{N} \sum_{(\mathcal{L}_1, \mathcal{L}_2) \in \Lambda} \langle f, e^{2\pi i \langle \mathcal{L}_1, x \rangle} h_j(x - \mathcal{L}_2) \rangle e^{2\pi i \langle \mathcal{L}_1, x \rangle} g_j(x - \mathcal{L}_2), \quad f \in L^2(\mathbb{R}^d).$$

Remark. Corollaries 3.1 and 3.2 are also true when $\mathcal{G}(\Lambda, g_1) \bigcup \mathcal{G}(\Lambda, g_2) \bigcup \cdots \bigcup \mathcal{G}(\Lambda, g_N)$ is a Gabor multi-frame generator for the subspace it generates.

3.2. Frames for shift invariant subspaces

Frames for shift invariant subspaces play an important role in wavelet and Gabor analysis. Let $\mathcal{K}$ be a lattice in $\mathcal{R}^d$. Recall that V is a *shift invariant subspace* (SIS for short) if V is a closed subspace of $L^2(\mathbb{R}^d)$ such that $T_\lambda(V) \subset V$ for every $\lambda \in \Lambda$. For each shift-invariant subspace V, there exists a unique measurable set $\Omega(V)$ which is called the *spectrum* of V. Moreover $\Omega(V)$ is the support of

$$G_\Phi(\gamma) := \sum_{j=1}^{N} \sum_{k \in \tilde{\mathcal{K}}} |\hat{\phi}_j(\gamma + k)|^2$$

whenever $\{T_k \phi_j : k \in \mathcal{K}, 1 \leqslant j \leqslant N\}$ is a frame for V, where $\tilde{\mathcal{K}}$ is the dual lattice of $\mathcal{K}$ and $\hat{\phi}$ is the Fourier transform of ϕ. The following is well-known:

Lemma 3.3. (i) $\{T_k h : k \in \mathcal{K}\}$ *is a normalized tight frame for a shift invariant subspace V if and only if* $G_h(\gamma) = \chi_{\Omega(V)}(\gamma)$.

(ii) $\{T_k h : k \in \mathcal{K}\}$ *is a frame for V if and only if G_h is bounded from below and above on its support.*

It is easy to check that if $\{T_k g : k \in \mathbb{Z}\}$ is a frame for V, then $\frac{\hat{g}(\gamma)}{\sqrt{G_g(\gamma)}} = \hat{S}^{-1/2} \hat{g}$, where $\hat{S}$ is the Fourier transform of the corresponding frame operator S and $\frac{\hat{g}(\gamma)}{\sqrt{G_g(\gamma)}}$ is defined to be zero when $G_g(\gamma) = 0$. Therefore, from Theorem 1.1, we have

Corollary 3.4. *Let V be a shift invariant subspace and $\{T_k g : k \in \mathcal{K}\}$ be a frame for V with $\Omega = supp(G_g)$. Then $\|\frac{\hat{g}}{\sqrt{G_g}} - \hat{g}\|$ minimizes $\|\hat{h} - \hat{g}\|$ over all $h \in V$ such that $G_h(\gamma) = \chi_\Omega(\gamma)$.*

By using Theorem 2.3, the above corollary can be generalized to the multi-frame case. For this the Gramian matrix is needed. Let $\Phi = (\phi_1, \dots, \phi_N) \subset L^2(\mathbb{R})$. Then the associated *Gramian matrix* is the $N \times N$ matrix $G_\Phi(\gamma) := (G_{ij}(\gamma))$, where

$$G_{ij}(\gamma) = \sum_{k \in \tilde{K}} \hat{\phi}_i(\gamma + k)\overline{\hat{\phi}_j(\gamma + k)}.$$

Let $M(\gamma)$ be the largest eigenvalue of $G(\gamma)$, $N(\gamma)$ be the smallest eigenvalue of $G(\gamma)$, and $N^+(\gamma)$ be the smallest non-zero eigenvalue of $G(\gamma)$. The following theorem of Ron and Shen characterizes the multi-frame generators in terms of the Gramian matrices:

Lemma 3.5 (Ron and Shen[RS2]). *Let V be a shift invariant subspace of $L^2(\mathbb{R})$ and $\Phi = (\phi_1, \dots, \phi_N) \subset V$. Then*

(i) *$\{T_k \phi_j : k \in \mathcal{K}, 1 \leqslant j \leqslant N\}$ is a frame for V if and only if $M(\gamma)$ and $1/N^+(\gamma)$ are essentially bounded on $\Omega(V)$.*
(ii) *$\{T_k \phi_j : k \in \mathcal{K}, 1 \leqslant j \leqslant N\}$ is a normalized tight frame for V if and only if G is a non-zero projection on $\Omega(V)$.*

Combining this with Theorem 2.3 we have

Corollary 3.6. *Let V be a shift invariant subspace of $L^2(\mathcal{R})$ and $\Phi = (\phi_1, \dots, \phi_N)$ be a frame generator for V. Write $h = (S^{-1/2}\phi_1, \dots, S^{-1/2}\phi_N)$ with S the associated frame operator. Then*

$$\sum_{j=1}^{N} \|S^{-1/2}\phi_j - \phi_j\|^2$$

minimizes

$$\sum_{j=1}^{N} \|\psi_j - \phi_j\|^2$$

over all $\Psi = (\psi_1, \dots, \psi_N) \subset V$ such that G_Ψ is a non-zero projection for a. e. $\gamma \in \Omega(V)$.

If, in addition, we require that $span\{T_k \phi_j : k \in \mathcal{K}\}$ and $span\{T_k \phi_{\updownarrow} : k \in \mathcal{K}\}$ are orthogonal for $j \neq \mathcal{L}$, then the Gramian matrix is diagonal and $\widehat{S^{-1/2}\phi_j} = \hat{\phi}_j \sqrt{G_\phi}$.

Thus

$$\sum_{j=1}^{N} \left|\left|\frac{\hat{\phi}_j}{\sqrt{G_\phi}} - \hat{\phi}_j\right|\right|^2$$

minimizes $\sum_{j=1}^{N} ||\psi_j - \phi_j||^2$ over all $\Psi = (\psi_1, \ldots, \psi_N) \subset V$ such that G_Ψ is a non-zero projection for a. e. $\gamma \in \Omega(V)$.

3.3. Finite group frames

A finite frame is a frame for a finite-dimensional space. Recently there has been a lot of interests in finite frames because of their usefulness in applications such as internet coding, wireless communication, quantum detection theory etc. An important class of finite frames are the frames obtained by a finite group action. Since we are dealing with finite-dimensional spaces we can assume that $\mathcal{H} = \mathbb{C}^n$. Let $\{\mathbf{v}_1, \ldots, \mathbf{v_m}\} \subset \mathcal{H}$. Then $\{\mathbf{v}_1, \ldots, \mathbf{v_m}\}$ is a frame if and only if its Gramian matrix $[\langle \mathbf{v_i}, \mathbf{v}_j \rangle]_{m\times m}$ has rank n, and it is a normalized tight frame if and only if its Gramian is a rank n projection.

Now let us consider a unitary representation τ of a finite group $\mathcal{G}$ on $\mathcal{H}$. Let $G(\tau, \mathbf{v}_1, \ldots, \mathbf{v_k})$ be the Gramian matrix of $\{\tau(g)\mathbf{v}_j : g \in \mathcal{G}, 1 \leqslant j \leqslant k\}$. Then we have the following:

Corollary 3.7. *Let $G(\tau, \mathbf{v}_1, \ldots, \mathbf{v}_k)$ be a rank n matrix and S be the associated frame operator. Then*

$$\sum_{j=1}^{k} ||\mathbf{v}_j - S^{-1/2}\mathbf{v_j}||^2 \leqslant \sum_{j=1}^{k} ||\mathbf{v}_j - \xi_j||^2$$

holds for all $\xi_1, \ldots, \xi_k \in \mathcal{H}$ such that $G(\tau, \mathbf{x}_1, \ldots, \mathbf{x_k})$ is a projection of rank n.

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